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Topological Groupoids

Ronson J. Warne
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a dissertation written by Ronson J. Warne entitled "Topological Groupoids." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Orville G. Harold, Major Professor

We have read this dissertation and recommend its acceptance:

E. Cohen, Edward D. Harris, T. A. Fisher, D. D. Lillian, W. H. Flecther

Accepted for the Council:
Dixie L. Thompson

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

May 27, 1959

To the Graduate Council:

I am submitting herewith a thesis written by Ronson J. Warne entitled "Topological Groupoids". I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

O. G. Harrold, Jr.
Major Professor

We have read this thesis and recommend its acceptance:

E. Cohen

Edward D. Harris

F. A. Ficken

D. D. Miller

W. H. Fletcher

Accepted for the Council:

Dale Wantling
Dean of the Graduate School

TOPOLOGICAL GROUPOIDS

A THESIS

Submitted to
The Graduate Council
of
The University of Tennessee
in
Partial Fulfillment of the Requirements
for the degree of
Doctor of Philosophy

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by

Ronson J. Warne

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CHAPTER I

INTRODUCTION

A groupoid is a set G in which a single valued product ab is defined for every pair of elements $a, b \in G$. If G is a groupoid and at the same time a Hausdorff topological space, and, moreover, the multiplication in the groupoid G is continuous in the topological space G, then G is called a topological groupoid. Our aim in this dissertation is two-fold: (1) to study topological groupoids for their own sake; (2) to investigate the relation of certain topological properties to associativity. We note, in relation to the first motif, that many authors have dealt with non-associative algebraic structures, i.e. Albert [1]*, Frink [4], Garrison [5], Etherington [2], Hausmann and Ore [7], and Stein [22].

If a is an element of a groupoid G, we will use the following notation: $\hat{a} = a$, $\widehat{a^2} = \hat{a} \cdot a$, $\widehat{a^3} = \widehat{a^2} \cdot a$, ..., $\widehat{a^n} = \widehat{a^{n-1}} \cdot a$, etc. An element a of a topological groupoid G is said to have property $\mathscr{a}$ if and only if (henceforth, we will use the common abbreviation iff introduced in [9]) $V(\widehat{a^n})[V(\widehat{a^r})V(a)] \cap [V(\widehat{a^n})V(\widehat{a^r})]V(a) \neq \square$ ($\square$ will denote the empty set) for every triplet of neighborhoods $V(\widehat{a^n})$ of $\widehat{a^n}$, $V(\widehat{a^r})$ of $\widehat{a^r}$, and $V(a)$ of a for all positive integers n and r. If $a \in G$ has property $\mathscr{a}$, all powers of a are unique. In Chapter 2, we will prove theorems relating to idempotents, ideals, etc. Many of these theorems are generalizations of theorems in mob

*Numbers in square brackets refer to numbers in the bibliography at the end of this paper.

theory. For example, we establish the following theorems: An elementwise bicompact groupoid G that has an element with property $\mathcal{a}$ contains at least one idempotent. If A is an algebraic subgroup of a topological groupoid G and $\bar{A}$ is bicompact, then $\bar{A}$ is a topological group. Every bicompact groupoid G has a unique minimal closed ideal, a minimal closed left ideal, a minimal closed right ideal, and a minimal closed subgroupoid. These concepts are defined in Chapter II (Definitions 2.3, 2.12, and 2.2). Every algebraic subgroup of a topological groupoid G is contained in a maximal algebraic subgroup which is a bicompact topological group if G is bicompact. If G is a bicompact groupoid, each minimal left ideal and each minimal right ideal is closed provided each element of K, the minimal closed ideal of G, is center associative [5]. The set of idempotents of a topological groupoid is closed.

Chapter II closes with a discussion of connected groupoids. If G is a bicompact, connected groupoid with a unit, the minimal closed ideal K of G is connected. If G is a groupoid with a unit, let I denote the set of elements of G such that $x \in I$ iff there exists a $y \in G$ so that $xy = yx = u$. If G is a connected groupoid such that I is bicompact and each element of I is center associative in G, then no element of I cuts I in G. If G is a bicompact, connected, non-simple [17] groupoid with a unit, no element of I cuts G provided a "certain" neighborhood condition is satisfied. If G is a connected groupoid with a zero, every left (right) ideal of G is connected provided G has a left (right) unit.

In Chapter III, we consider bicompact groupoids G with a zero and a unit which are irreducibly connected between 0 and u and which obey the cancellation law. $xc = yc$ or $cx = cy$ for $x, y, c \in G$ with $c \neq 0$ implies $x = y$. If x, y, and $c \in G$, $x < y$, and $c \neq 0$, then $xc < yc$ and $cx < cy$. If $x \neq u$ and x has property $\mathcal{a}$, then $\{x^n : n \in I\}$, where I represents the positive integers, converges to 0. Our main result states that if every element of G has property $\mathcal{a}$, then there exists a function f from G onto the unit interval $[0, 1]$ of real numbers under the usual topology and the usual multiplication, which is an isomorphism as well as an order preserving homeomorphism. We note that the unit in the above theorem may be replaced by any center associative idempotent. In particular, any continuous multiplication on $[0, 1]$ as a space is the usual multiplication of real numbers if the multiplication obeys the cancellation law, 0 acts as a zero, 1 acts as a unit, and every element of $[0, 1]$ has property $\mathcal{a}$. We next study the "associative-like" properties of G. Using these properties, we establish some of the results of mob theory for this type of groupoid.

In Chapter IV, we are concerned with monothetic groupoids, topological groupoids with a zero, and quasi-groups. A topological groupoid G is monothetic iff there exists $a \in G$ such that the collection of all finite "words" in a is dense in G. We call a a generator of G. We consider the net $\{b_n : n \in I\}$ (see definition 4.3). This net clusters at each idempotent of G. If G is bicompact, $\bigcap_{n=1}^{\infty} \overline{\{b_i : i \geqq n\}}$ is a closed ideal of G such that $A \subset G$, $A = \bar{A}$, and $AA = A$ imply $A \subset \bigcap_{n=1}^{\infty} \overline{\{b_i : i \geqq n\}}$. Let G be a monothetic groupoid with a unit and suppose that $G = P \cup Q$ where P and Q are disjoint, closed, and non-empty. Then P and Q each contain infinitely many finite "words" in any generator a of G. We give necessary and sufficient conditions for a monothetic groupoid to be a monothetic semi-group. We next consider the general theory of topological groupoids G with a zero. If $a \in G$ and a^n is unique for all n, then a is called a nilpotent provided $a^n \to 0$. A nil left (right, two-sided) ideal of G is a left (right, two-sided) ideal of G which consists entirely of nilpotent elements. If G is elementwise bicompact, every right (left, two-sided) ideal A of G is either a nil ideal or contains non-zero idempotents provided every element of A has property $\mathcal{A}$. The groupoid G is said to be an N-groupoid iff its nilpotent elements form an open set. If every element of G has property A, then G is an N-groupoid provided there exists a neighborhood V_0 of 0 which consists entirely of nilpotent elements. The radical (Def. 4.9) of a bicompact N-groupoid is open provided each nilpotent element is right associative [5]. We show that any bicompact subset of G is bounded. Finally, we consider the special groupoids with a zero mentioned in Chapter III. We discuss the concepts of nil ideal, radical, and "arbitrarily small" bicompact ideal neighborhoods of 0 in these groupoids. We close this chapter with an application of topological techniques to the theory of quasi-groups. A groupoid G is called a quasi-group iff for any two elements a and b in G, there exists exactly one $x \in G$ such that $ax = b$ and exactly one $y \in G$ such that $ya = b$. It is well known that a finite groupoid that satisfies the cancellation law ($ax = ay$ or $xa = ya$ implies $x = y$ for all $a, x, y \in G$) is a quasi-group. We give conditions for a bicompact cancellation groupoid to be a quasi-group.

CHAPTER II

GENERALITIES ON TOPOLOGICAL GROUPOIDS

Definition 2.1. A groupoid is a set G in which a single valued product ab is defined for every pair of elements $a, b \in G$.

Definition 2.2. A subgroupoid in a groupoid G is a non-void set T contained in G such that $TT \subset T$.

Definition 2.3. (i) A left (right) ideal of G is a non-void set T contained in G such that $GT \subset T$ $(TG \subset T)$; (ii) a two-sided ideal of G is a subset of G which is both a left ideal and a right ideal.

Definition 2.4. An element e in G is called an idempotent iff $e^2 = e$.

Definition 2.5. An element $0 \in G$ is termed a zero iff $0x = 0 = x0$ for all $x \in G$.

Clearly, if a zero exists in G it is unique and is an idempotent.

Definition 2.6. An element $u \in G$ is termed an identity of G iff $ux = x = xu$ for all $x \in G$.

Clearly, if an identity exists in G, it is unique and is an idempotent.

Definition 2.7. A groupoid G in which $(ab)c = a(bc)$ for all a, b, and c in G is called a semi-group.

Definition 2.8. If G is a groupoid and at the same time a Hausdorff topological space and, moreover, the multiplication in the groupoid G is continuous in both variables with respect to the topological space G, then G is called a topological groupoid.

Definition 2.9. A topological groupoid in which the multiplication is associative is called a <u>topological semi-group</u> or a <u>mob</u>.

If T_1 and T_2 are bicompact subsets of a topological groupoid G, then T_1T_2 is also bicompact. By Tycnonoff's theorem $T_1 \times T_2$ is bicompact and T_1T_2 is the continuous image of $T_1 \times T_2$. It thus follows that if G is a bicompact groupoid then aG, Ga, G^2, a^2G and Ga^2 are bicompact for a in G.

We also observe that if T_1 and T_2 are connected subsets of a topological groupoid G, then T_1T_2 is connected.

A subgroupoid G_1 of a topological groupoid G is itself a topological groupoid under the relative topology.

The following theorem gives a statement of the associative law in topological groupoids in terms of neighborhoods.

<u>Theorem</u> 2.1. <u>A necessary and sufficient condition for a topological groupoid</u> G <u>to be a topological semi-group is that</u> $[V(a)V(b)]V(c) \cap V(a)[V(b)V(c)] \neq \square$ <u>for all triplets of neighborhoods</u> $V(a)$, $V(b)$, <u>and</u> $V(c)$ of a, b, <u>and</u> c <u>respectively for all</u> a, b, c <u>in</u> G.

<u>Proof:</u> Suppose the condition is satisfied and there exist elements a, b, and c of G such that $(ab)c \neq a(bc)$. Then one can find neighborhoods $V(a)$ of a, $V(b)$ of b, and $V(c)$ of c such that $[V(a)V(b)]V(c) \cap V(a)[V(b)V(c)] = \square$. But this is a contradiction.

On the other hand, the condition is clearly satisfied for a topological semi-group.

We will denote the topological closure of a subset S of G by $\bar{S}$.

<u>Lemma</u> 2.1. <u>Let</u> G <u>be a topological groupoid and let</u> S <u>be a subsemi-group of</u> G. <u>Then</u> $\bar{S}$ <u>is a closed subsemi-group of</u> G.

<u>Proof:</u> Suppose $a \in S$ and $b \in S$. Let $V(ab)$ be an arbi-

trary neighborhood of ab. Then there exists neighborhoods $V(a)$ and $V(b)$ of a and b respectively such that $V(a)V(b) \subset V(ab)$. But, there exists $s_1 \in S$ and $s_2 \in S$ such that $s_1 \in V(a)$ and $s_2 \in V(b)$. Thus, $s_1 s_2 \in V(ab) \cap S$ and $ab \in \bar{S}$.

Next, suppose there exists a, b, and $c \in \bar{S}$ such that $(ab)c \neq a(bc)$. Then one can find neighborhoods $V(a)$ of a, $V(b)$ of b, and $V(c)$ of c such that $[V(a)V(b)]V(c) \cap V(a)[V(b)V(c)] = \square$. But, there exist elements s_1, s_2, and s_3 of S such that $s_1 \in V(a)$, $s_2 \in V(b)$, and $s_3 \in V(c)$. Thus $s_1 s_2 s_3 \in [V(a)V(b)]V(c) \cap V(a)[V(b)V(c)]$. Hence, we have a contradiction and $(ab)c = a(bc)$ for all a, b, and c in $\bar{S}$.

Analogously, we have

<u>Lemma</u> 2.2. <u>Let</u> G <u>be a topological groupoid and let</u> A <u>be right (left, two-sided) ideal of</u> G. <u>Then</u> $\bar{A}$ <u>is a closed right (left, two-sided) ideal of</u> G.

Definition 2.10. We will make use of the following <u>notation</u>: $\hat{a} = a$, $\hat{a^2} = \hat{a} \cdot a$, $\hat{a^3} = \hat{a^2} \cdot a$, ... $\hat{a^n} = \hat{a^{n-1}} \cdot a$.

Definition 2.11. An element a of a topological groupoid G is said to have Property $\mathcal{Q}$ iff $[V(\hat{a^n})V(\hat{a^r})]V(a) \cap V(\hat{a^n})[V(\hat{a^r})V(a)] \neq \square$ for every triplet of neighborhoods $V(\hat{a^n})$ of $\hat{a^n}$, $V(\hat{a^r})$ of $\hat{a^r}$ and $V(a)$ of a for all positive integers n and r.

Example 2.1. An example of a finite topological groupoid which is not a topological semi-group although every element has property $\mathcal{Q}$ is given by the set $\{1, 2, 3\}$ with the following multiplication table and the discrete topology:

	1	2	3
1	1	3	x
2	x	2	1
3	x	x	3

x may be 1, 2, or 3

Lemma 2.3. Let G be a topological groupoid. If an element a of G has property $\mathcal{A}$, then all the powers of a are unique.

Proof: It is first shown that $(\widehat{a^n}\widehat{a^r})a = \widehat{a^n}(\widehat{a^r}a)$ for all positive integers n and r. Suppose this were not the case. Then there exists positive integers n and r such that $(\widehat{a^n}\widehat{a^r})a \neq \widehat{a^n}(\widehat{a^r}a)$. Thus, there exist neighborhoods $V(\widehat{a^n})$, $V(\widehat{a^r})$, and $V(a)$ of $\widehat{a^n}$, $\widehat{a^r}$, and a respectively such that $[V(\widehat{a^n})V(\widehat{a^r})]V(a) \cap V(\widehat{a^n})[V(\widehat{a^r})V(a)] = \square$. But this contradicts property $\mathcal{A}$. Thus, $(\widehat{a^n}\widehat{a^r})a = \widehat{a^n}(\widehat{a^r}a)$ for all positive integers n and r. It is next shown that $\widehat{a^{n+m}} = \widehat{a^n} \cdot \widehat{a^m}$ for all positive integers m and n. This is true for $m = 1$ by definition. We will assume it is true for $m = r$ and prove it is true for $m = r + 1$. Now, $\widehat{a^m} \cdot \widehat{a^{r+1}} = \widehat{a^m}(\widehat{a^r}a) = (\widehat{a^m}\widehat{a^r})a = \widehat{a^{m+r}}a = \widehat{a^{m+r+1}}$. Thus, all the powers of a are unique.

Definition 2.12. A topological groupoid G is elementwise bicompact iff for every $a \in G$, such that all the powers of a are unique, the set $\{a^n : n \in I\}$, where I is the positive integers and a^n is the unique n^{th} power of a, is contained in a bicompact subset of G.

Example 2.2. An example of an elementwise bicompact groupoid which is is not bicompact. Let $G = (0, 1)$ with the usual topology. Define $xy = 1/2$ for all x and y in G. Let $a \in G$, say $a < 1/2$. Then $\{a^n : n \in I\} \subset [a, 1/2]$.

Theorem 2.2. An elementwise bicompact groupoid G that has an element with property $\mathfrak{a}$ contains at least one idempotent.

Proof. Suppose $a \in G$ and a has property $\mathfrak{a}$. Then, by Lemma 2.3, the powers of a are unique. As is customary denote these powers by a^n, etc., where n is a positive integer. Then $\{a^n : n \in I\}$, where I represents the positive integers, is a topological semi-group. Hence $\overline{\{a^n : n \in I\}}$ is a topological semi-group by Lemma 2.1. Let $A_\gamma = \{a^i : i \geq \gamma\}$ and $U = \{\overline{A_\gamma : \gamma = 1, 2, \ldots}\}$. Since U has the finite intersection property and $\{a^n : n \in I\}$ is bicompact by virtue of the elementwise bicompactness of G, $D = \bigcap_{\gamma=1}^{\infty} \bar{A} \neq \square$. It will be shown that D is a commutative closed group. Since $D \subset \overline{\{a^n : n \in I\}}$, D obeys the associative law. Hence it is clear that D is a commutative closed semi-group. It remains to show that D forms a group. To prove this it is sufficient to show that $xD = D$ for all $x \in D$. Suppose there exists y in D such that $yD \subsetneqq D$. Then there is a z in D such that $z \bar{\in} yD$, that is, $z \neq yx_\alpha$ for every x_α in D. Therefore, there exist neighborhoods $V_\alpha(y)$ of y, $V(x_\alpha)$ of x_α, and $V_\alpha(z)$ of z such that $V_\alpha(z) \cap V_\alpha(y)V(x_\alpha) = \square$. Since $\bigcup_{x_\alpha \in D} V(x_\alpha) \supset D$ and D is bicompact we can choose a finite subcollection $V(x_{\alpha_i})$ $(i = 1, 2, \ldots n)$ of the $V(x_\alpha)$ so that $\bigcup_{i=1}^{n} V(x_{\alpha_i}) \supset D$. Let $V(y)$ and $V(z)$ be neighborhoods of y and z respectively such that $V(y) \subset \bigcap_{i=1}^{n} V_{\alpha_i}(y)$ and $V(z) \subset \bigcap_{i=1}^{n} V_{\alpha_i}(z)$. Let $\bigcup_{i=1}^{n} V(x_{\alpha_i}) = Q$. Q is an open set containing D and $V(z) \cap V(y)Q = \square$. Since $y \in D$, there is an integer $u \geq 1$ so that $a^u \in V(y)$. Since $z \in D$, there exist integers v_i such that $v_i > u$, $v_{i+1} > v_i$ $(i = 1, 2, \ldots)$, and $a^{v_i} \in V(z)$ for every v_i. Put $t_i = v_i - u \geq 1$ and $A^{(t_c)} = \{a^{t_j} : j = c, c+1, \ldots\}$. Then

$\bigcap_{c=1}^{\infty} \overline{A^{(t_c)}} \neq \Box$ and it is easily shown that $\bigcap_{c=1}^{\infty} \overline{A^{(t_c)}} \subset D$. Choose an element $d \in \bigcap_{c=1}^{\infty} \overline{A^{(t_c)}}$. Then since Q is an open set containing D, there exists a neighborhood $V(d)$ of d contained in Q, and since $d \in \bigcap_{c=1}^{\infty} \overline{A^{(t_c)}}$, there exists an integer t_k such that $a^{t_k} \in V(d)$. Thus, $a^{v_k} \in V(z)$ and $a^{v_k} = a^{u+t_k} = a^u a^{t_k} \in V(y)V(d) \subset V(y)Q$. This contradicts the fact that $V(z) \cap V(y)Q = \Box$. Hence, we obtain $xD = D$ for all $x \in D$ and D is a group. If e is the identity of D, $e^2 = e$.

Corollary 2.1. *A necessary and sufficient condition that an element-wise bicompact groupoid G have an element a such that all powers of a are unique is that G have an idempotent element.*

Lemma 2.4. *Let G be a topological groupoid and let A be a subset of G. Let $g : A \longrightarrow G$ be such a function that (1) $\overline{g(A)}$ is compact. (2) For some $c \in G$ and all $x \in A$, $x\,g(x) = c$. (3) If $xy = c$, then $y = g(x)$. Then g is a map (a continuous function).*

Proof: If not, then for some $B \subset A$ and some $x \in \overline{B}$, we have $g(x) \in G \setminus \overline{g(B)}$. Then $x\,\overline{g(B)} \subset G \setminus c$. Hence we may obtain an open set U including x with $U\,\overline{g(B)} \subset G \setminus c$ [18]. If $a \in U \cap B$, then $ag(a) = c$ and we have a contradiction.

Theorem 2.3. *Let A be an algebraic subgroup of the topological groupoid G. If $\overline{A}$ is compact then $\overline{A}$ is a topological subgroup.*

Proof: $\overline{A}$ is a submob by Lemma 2.1. We will next show that $\overline{A}$ is a subgroup. It will be sufficient to show that $x\overline{A} = \overline{A}$ for all $x \in \overline{A}$ and $\overline{A}x = \overline{A}$ for all $x \in \overline{A}$. Suppose there is a $y \in \overline{A}$ such that $y\overline{A} \subsetneq \overline{A}$. Then there exists $z \in \overline{A}$ such that $z \neq yx_\alpha$ for all x_α in $\overline{A}$. Thus one may find neighborhoods $V_\alpha(z)$ of z, $V_\alpha(y)$ of y, and $V(x_\alpha)$ of x_α for all x_α in $\overline{A}$ so that $V_\alpha(z) \cap V_\alpha(y)V(x_\alpha) = \Box$. Since $\overline{A}$ is compact, there exists a finite subcollection $\{V(x_{\alpha_i}): i = 1, 2, \ldots k\}$ of the $V(x_\alpha)$'s so that $\bigcup_{i=1}^{k} V(x_{\alpha_i}) \supset \bar{A}$. Let $Q = \bigcup_{i=1}^{k} V(x_{\alpha_i})$. Now one may find neighborhoods $V(z)$ and $V(y)$ of z and y respectively so that $V(z) \cap V(y)Q = \Box$. Now, there exist elements a and b of A such that $a \in V(z)$ and $b \in V(y)$. But, $a = b(b^{-1}a) \in V(y)Q$ and we have a contradiction. Thus $x\bar{A} = \bar{A}$ for all $x \in \bar{A}$. Similarly $\bar{A}x = \bar{A}$ for all $x \in \bar{A}$. Thus, $\bar{A}$ is a group.

It follows immediately from Lemma 2.4 that the function $x \rightarrow x^{-1}$ where $x \in \bar{A}$ is a continuous function of $\bar{A}$ onto itself. Thus, $\bar{A}$ is a topological subgroup.

Definition 2.13, [9]. A <u>directed set</u> $(D, >)$ is a non-empty set D together with a binary relation $>$ satisfying (i) if m, n, and p belong to D and if $m > n$ and $n > p$, then $m > p$. (ii) if $m \in D$, then $m > m$. (iii) if $m \in D$ and $n \in D$ then for some $p \in D$, $p > m$ and $p > n$.

A <u>net</u> is a function v on a directed set D to a set X. If X is a topological space, we say (i) $a \in X$ is a <u>cluster point</u> of the net v if for any open set $\mathcal{O}$ containing a, and any $d \in D$, there is a $d' \in D$ such that $d' > d$ and $v(d') \in \mathcal{O}$; (ii) $a \in x$ is a <u>point of convergence</u> of the net v if given any open set $\mathcal{O}$ containing a, there is a $d \in D$ such that if $d' \in D$ and $d' > d$, then $v(d') \in \mathcal{O}$.

Definition 2.14. A <u>minimal left (right, two-sided) ideal</u> L is a left (right, two-sided) ideal such that if A is any left (right, two-sided) ideal and $A \subset L$, then $A = L$.

We note that a minimal two-sided ideal of a groupoid G is unique.

For let A and B be minimal two sided ideals of G . Then $A \cap B \neq \square$ since $AB \subset A \cap B$. Thus, $A \cap B$ is an ideal and $A = B = A \cap B$.

Theorem 2.4. *Let G be a topological groupoid and suppose $a \in G$ has property $\mathcal{a}$. Then, if $\overline{\{a^n : n \in I\}}$ (note that the powers of a are unique) is compact, the set of all cluster points of the net v defined by $v(n) = a^n$ forms a topological group and is the minimal two sided ideal of $\overline{\{a^n : n \in I\}}$.*

Proof. The set $D = \bigcap_{\gamma=1}^{\infty} \overline{\{a^i : i \geq \gamma\}}$ is the set of all cluster points of v . We may show, as in the proof of Theorem 2.2, that D is a group. Since D is compact, it is a topological group by Theorem 2.3. Next, we show that D is a two-sided ideal of the mob $\overline{\{a^n : n \in I\}}$. Let $y \in \overline{\{a^n : n \in I\}}$ and $x \in D$. Suppose $O(xy)$ is an arbitrary neighborhood of xy . Then one can find neighborhoods $O(x)$ of x and $O(y)$ of y such that $O(x)O(y) \subset O(xy)$. But, there exists $a^{\gamma+1_\gamma} \in O(x)$ for $\gamma = 1, 2, \ldots$ since $x \in D$. There also exists $a^n \in O(y)$. Hence $a^{\gamma+1_\gamma+n} \in O(xy)$ for $\gamma = 1, 2, \ldots$. Thus $xy \in D$. Similarly $yx \in D$. Thus D is a two-sided ideal of $\overline{\{a^n : n \in I\}}$. Since D is a group, it cannot contain properly any two-sided ideal. For suppose A is a two-sided ideal of D . Let $y \in D$ and $a \in A$. Then $ya = b \in A$. But, $y = ba^{-1} \in A$. Hence $A = D$. Thus D is a minimal two-sided ideal of $\overline{\{a^n : n \in I\}}$. By the remark above, such an ideal is unique.

Theorem 2.5. *Every compact groupoid G has a minimal closed subgroupoid.*

Proof. Let $\mathcal{a}$ be the family of all closed subgroupoids of G . Let $\eta = \{A_\alpha : \alpha \in \Lambda\}$ be any nest [9] in $\mathcal{a}$. Since η has the finite intersection property, $\bigcap_{\alpha \in \Lambda} A_\alpha \neq \square$ and hence is a closed groupoid. But, $\bigcap_{\alpha \in \Lambda} A_\alpha \subset A_\alpha$ for all $\alpha \in \Lambda$. Thus, $\mathcal{A}$ has a minimal member by the minimal principle [9].

Theorem 2.6. <u>Every compact groupoid G contains the unique minimal closed ideal.</u>

Proof: Let $\mathcal{A}$ be the family of all closed ideals of G. Let $\eta = \{B_\gamma : \gamma \in \Lambda\}$ be any nest in $\mathcal{A}$. Then, $\bigcap_{\gamma \in \Lambda} B_\gamma \neq \square$ and hence is a closed ideal of G. But, $\bigcap_{\gamma \in \Lambda} B_\gamma \subset B_\gamma$ for every $\gamma \in \Lambda$. Thus, $\mathcal{A}$ has a minimal member. The uniqueness was established previously.

Theorem 2.7. <u>Every compact groupoid G has a minimal closed left ideal and a minimal closed right ideal.</u>

Proof: The proof of this theorem is similar to that given for Theorem 2.6.

Theorem 2.8. <u>Any algebraic subgroup of a topological groupoid G is contained in a maximal algebraic subgroup.</u>

Proof: Let A be any algebraic subgroup of the topological groupoid G. Let $\mathcal{A}$ be the family of all subgroups of G containing A partially ordered by inclusion. Let $\eta = \{A_\alpha : \alpha \in \Lambda\}$ be any chain in $\mathcal{A}$. Let us consider $B = \bigcup \{A_\alpha : \alpha \in \Lambda\}$. Let $x, y \in B$. Then $x \in A_\alpha$ and $y \in A_\beta$ for some α and $\beta \in \Lambda$. We may assume $A_\alpha \subset A_\beta$. Thus $xy \in A_\beta \subset B$. Suppose x, y, and z are in B. Then $x \in A_\alpha$, $y \in A_\beta$, and $z \in A_\gamma$ for some α, β, and $\gamma \in \Lambda$. We may assume $A_\alpha \subset A_\beta \subset A_\gamma$. Hence x, y, and z are in A_γ and $(xy)z = x(yz)$. Let e be the identity of A. Then e is the identity of A_α for all $\alpha \in \Lambda$ and hence e is an identity of B. Let $x \in B$. Then $x \in A_\alpha$ for some $\alpha \in \Lambda$. Therefore $x^{-1} \in A_\alpha \subset B$

and $xx^{-1} = e = x^{-1}x$. Thus, B is a group containing A and $B \supset A_\alpha$ for all $\alpha \in \wedge$. Hence, by Zorn's lemma [9], $\mathcal{a}$ has a maximal member. Now, suppose B' is an algebraic subgroup of G containing the maximal member C of A . Then, $B' \supset A$ and $B' = C$.

Theorem 2.9. *If G is a compact topological groupoid, each maximal algebraic subgroup of G is a compact topological group.*

Proof: If A is an algebraic subgroup, $\bar{A}$ is compact and hence is a topological group by Theorem 2.3. Since A is maximal, $A = \bar{A}$.

Definition 2.15. We shall call an element c of a groupoid G *center associative* in G if $x(cy) = (xc)y$ for all $x, y \in G$. [5]

Theorem 2.10. *If G is a bicompact groupoid, each minimal left ideal and each minimal right ideal of G is closed provided each element of K, the minimal closed ideal of G, is center associative in G.*

Proof: Let L be a minimal left ideal of G . $Ka \subset L$ for $a \in L$ since L is a left ideal. $b(Ka) \subset (bK)a \subset Ka$ since every element of K is center associative in G . Thus, Ka is a left ideal and $Ka = L$ because of the minimality of L . Hence, L is closed. Similarly, we show that if R is a minimal right ideal of G , aK is a right ideal contained in R for $a \in R$. Hence $aK = R$ and R is closed.

Theorem 2.11. *The set of idempotents E of a topological groupoid is closed.*

Proof: If $\bar{E} = \Box$, the result is trivial. Suppose $\bar{E} \neq \Box$ and $p \in \bar{E}$. If $p \neq p^2$, there exists a neighborhood V(p) of p such that $[V(p)]^2 \cap V(p) = \Box$. Since $p \in \bar{E}$, $V(p) \cap E \neq \Box$, i.e., an idempotent f exists in V(p) . This contradicts $[V(p)]^2 \cap V(p) = \Box$.

Thus, $p^2 = p$, $p \in E$, and E is closed.

We next discuss briefly the concept of connectivity as related to topological groupoids. This will be discussed more fully in Chapter III.

Theorem 2.12. *Let G be a compact, connected groupoid with a unit. Then the minimal closed ideal K of G is connected.*

Proof: Let C be a component of K. Then, $C = uC \subset GC \subset GK \subset K$. Since GC is a connected subset of K, $GC = C$. Similarly $CG = C$. Since K is closed, C is closed. Thus, $C = K$ from the minimality of K.

Definition 2.17. If G is a topological groupoid with a unit, let I denote the set of elements such that $x \in I$ iff there exists a $y \in G$ so that $xy = yx = u$.

Definition 2.18. Let us agree that $A|B$ means $\bar{A} \cap B = \square = A \cap \bar{B}$ i.e., A and B are separated. Let us say that P cuts Q in G if $G - P = U \cup V$, $U|V$, $Q \cap U \neq \square$, and $Q \cap V \neq \square$.

We next state a theorem due to Wallace [18, Theorem 1].

Lemma 2.4. *Let X be a connected Hausdorff space and let C be a compact subset of X with at least two points. Then C has at least two points, neither of which cuts C in X.*

Theorem 2.13. *If G is a connected groupoid such that I is compact and each element of I is center associative in G, then no element of I cuts I in G.*

Proof: If $I = u$, the result follows immediately. Thus we may assume I is non-degenerate. We first note that I is a group since each element of I is center associative. Let $x \in I$ and $x \neq u$. Define $f(s) = x \cdot s$ and $g(s) = x^{-1}s$ for all $s \in G$. Using the center associative property, it is easily seen that f is a homeomorphism of G onto itself. Since I is a groupoid and each element of I is center associative, $f(I) = I$. Now assume u cuts I in G. Then since $f(u) = x$, each point of I cuts I in G. But, this contradicts lemma 2.4. Hence, u does not cut I in G. If $x \in I$ cuts I in G, then $g(x) = u$ cuts I in G. Thus, no $x \in I$ cuts I in G.

Definition 2.19. A groupoid G is said to be simple [17] iff G contains no proper closed ideals.

Theorem 2.14. *If G is a compact, connected, non simple groupoid with a unit such that $V(y)[V(x)V(s)] \cap [V(y)V(x)]V(s) \neq \Box$ for every triplet of neighborhoods $V(y)$, $V(x)$, and $V(s)$ of y, x, and s respectively for all y, x, and $s \in G$ such that $yx = u$, then no element of I cuts G.*

Proof: Since G is non simple, $G \neq K$ and hence $u \in G \setminus K$. We note that K is connected by virtue of Theorem 2.12. If u cuts G, then $G = A \cup B$, $A \cap B = u$, $A \setminus u \neq \Box$, $B \setminus u \neq \Box$, and A and B are each continua [6]. We may assume that $K \subset A \setminus u$. If $x \in B$, then since $xK \subset K$, the connected set xA has a point in both A and B. Therefore, $u \in xA$ and x has a right inverse y', i.e., $xy' = u$. Similarly, x has a left inverse. Utilizing the hypothesis, reasoning as in Theorem 2.1, and applying elementary algebraic techniques one sees easily that x has a unique inverse $\theta(x)$. Let us consider the maps $f, g : G \to G$ where $f(s) = x \cdot s$ and $g(s) = \theta(x) \cdot s$. Utilizing the hypothesis, one sees that $f(g(s)) = s$ and $g(f(s)) = s$ and if $s \in G$, $s = f(\theta(x) \cdot s)$. Clearly the maps f and g are continuous. Thus, f is a homeomorphism of G onto itself. Hence, since u cuts G, $x = f(u)$ cuts G. Therefore, each point of B is a cut point of

G . But B has at least two non cut points of itself. Let $p \in B \setminus u$ be one of them. Then it is easily shown that p cannot cut G . This is a contradiction and hence u cannot cut G . Now, if $x \in I$ cuts G, then $g(x) = u$ cuts G . Hence no x in I cuts G as asserted.

Example 2.3. Let $[0,1]$ be the unit interval of real numbers with the usual topology. Define the following multiplication on $[0,1]$: $a \circ b = a^2 b$ where the multiplication on the right is the usual multiplication of real numbers. $[0,1]$ with this multiplication is clearly a topological groupoid but not a topological semi-group. It is non-simple, connected, satisfies the neighborhood condition of the above theorem, 1 is a left unit, and I (with respect to 1) $= 1$. Obviously, no element of I cuts $[0,1]$.

<u>Theorem</u> 2.15. <u>Let</u> G <u>be a connected groupoid with a zero</u>.

(1) <u>If</u> G <u>has a left unit, every left ideal of</u> G <u>is connected</u>.

(2) <u>If</u> G <u>has a right unit, every right ideal of</u> G <u>is connected</u>.

(3) <u>If</u> G <u>has a right unit or a left unit, every ideal of</u> G <u>is connected</u>.

<u>Proof</u>: (1) Let L be a left ideal of G . If $x \in L$, $x \in Gx \subset L$. Hence, $L = \bigcup_{x \in L} Gx$ and $0 \in Gx$ for all $x \in L$. Hence L is connected since each Gx is connected.

(2) Let R be a right ideal of G . If $x \in R$, $x \in xG \subset R$. Hence $R = \bigcup_{x \in R} xG$ and $0 \in xG$ for all $x \in R$. Hence R is connected since each xG is connected.

(3) The proof of (3) is similar to those given in (1) and (2).

<u>Remark</u>. We note that 0 could have been replaced by left zero in (1) and by right zero in (2).

CHAPTER III

GROUPOIDS IRREDUCIBLY CONNECTED BETWEEN TWO IDEMPOTENTS

Definition 3.1. A space S is <u>irreducibly connected</u> between two points a and b if and only if it is connected and is the only connected subset of itself containing both a and b.

If such a space is Hausdorff every point different from a and b is a cut point, separating the space into exactly two components, one containing a and the other containing b [21]. We can introduce a linear order relation in S by defining $y < x$ if $y \in C$, C the component of b in $S - x$ [21]. Under this order relation, $b \leq x \leq a$ for every $x \in S$.

Moreover, for any $x \in S$, the set $A(x) = \{y : y \in S, y > x\}$ is precisely the component of a in $S - x$ and the set $B(x) = \{y : y \in S, y < x\}$ is precisely the component of b in $S - x$ [21]. It is clear that we can define a reverse order relation in S such that $a \leq x \leq b$ for every $x \in S$, by replacing a for b in the order relation.

This order relation induces a topology in S, the so-called order topology, with basis elements of the form $W = \{y : s < y < t \text{ and } s, t \in S\}$ [21]. Since we wish to use the order topology in this section, we prove that if S is compact, the order topology is equivalent to the original topology of S.

Consider $f : (S, U) \longrightarrow (S, \Upsilon)$ where U is the original topology of S, Υ is the order topology, and f is the identity function. Since $\Upsilon \subset U$ [21] and since S is compact, f is a homeomorphism extablishing our assertion that the topologies are equivalent.

Now, suppose S is a topological groupoid with a zero and a unit and S is irreducibly connected between 0 and u. We shall choose one of the two order relations we have defined in S so that $0 \leq x \leq u$.

We shall denote $[s, t] = \{y \in S, s \leq y \leq t\}$. The set $[s,t]$ is compact if S is compact and is irreducibly connected between s and t [21].

It follows easily that

Lemma 3.1. *Let S be a connected Hausdorff space and C a component of $S - x$, $x \in S$. If T is a connected subset of S such that $T \cap C \neq \square \neq T \cap (S - C)$, then $x \in T$. Further, if $S - x = A \cup B$, A and B are separated, and if $T \not\subset A$ and $T \not\subset B$, then $x \in T$.*

Lemma 3.2. *Let S be a compact Hausdorff space irreducibly connected between two points. If T is a connected set contained in S and if s and t are two points in T, then $[s, t] \subset T$.*

Proof. Let a and b be the two non cut points of S. Thus, we have $b \leq s < t \leq a$. Let $y \in [s,t]$ such that $y \neq s, t$. Then, $b \leq s < y < t \leq a$, which implies y is a cut point of S. Hence $S - y = A \cup B$ where A and B are separated. Since A and B are connected [6], neither A nor B contains both s and t. Otherwise, $[s, t]$ is not irreducibly connected between s and t. Thus, T is a connected set meeting both A and B. By Lemma 3.1, $y \in T$. Hence $[s, t] \subset T$.

Lemma 3.3. *Let S be a compact Hausdorff topological space which is irreducibly connected between two points a and b. Let A and B be subsets of S such that*

(1) $S = A \cup B$

(2) $A \cap B = \square$

(3) $A \neq \square$, $B \neq \square$

(4) *if* $\alpha \in A$ *and* $\beta \in B$, *then* $\alpha < \beta$.

Then there is one (and only one) $\gamma \in S$ *such that* $\alpha \leq \gamma$ *for all* $\alpha \in A$ *and* $\gamma \leq \beta$ *for all* $\beta \in B$.

Proof. Since S is connected, $\bar{A} \cap B \neq \square$ or $A \cap \bar{B} \neq \square$. Suppose $\bar{A} \cap B \neq \square$. Let $\gamma \in \bar{A} \cap B$. Then $\gamma \in B$. Suppose there exists $\beta \in B$ such that $\beta < \gamma$. Then there is a basis element $U(\gamma)$ containing γ such that $U(\gamma) \cap A = \square$. This contradicts the fact that $\gamma \in \bar{A}$. Hence, $\alpha < \gamma$ for all $\alpha \in A$ and $\beta \geq \gamma$ for all $\beta \in B$. Next assume $A \cap \bar{B} \neq \square$ and let $\gamma \in A \cap \bar{B}$. Hence, $\gamma \in A$. Suppose there exists $\alpha \in A$ such that $\alpha > \gamma$. Thus there exists a basis element $U(\gamma)$ containing γ such that $U(\gamma) \cap B = \square$ contradicting the fact that $\gamma \in \bar{B}$. Hence $\alpha \leq \gamma$ for all $\alpha \in A$ and $\beta > \gamma$ for all $\beta \in B$. Suppose there are two elements γ_1 and γ_2 such that $\alpha \leq \gamma_1, \gamma_2$ for all $\alpha \in A$ and $\beta \geq \gamma_1, \gamma_2$ for all $\beta \in B$. Suppose $\gamma_1 < \gamma_2$. Then there exists γ_3 such that $\gamma_1 < \gamma_3 < \gamma_2$. Thus $\gamma_3 < \gamma_2$ implies $\gamma_3 \in A$ whereas $\gamma_1 < \gamma_3$ implies $\gamma_3 \in B$. This contradicts 2.

Corollary 3.1. *Under the hypothesis of Lemma* 3.3, *either* A *contains a largest element or* B *contains a smallest element.*

Proof. If $\gamma \in A$, γ is the largest element in A. If $\gamma \in B$ γ is the smallest element in B. By (1) one of these cases must occur, while (2) implies both cannot occur together.

Definition 3.2. Let S be a compact Hausdorff space irreducibly connected between two points. Let E be a subset of S. If there is a $y \in S$ such that $x \geq y$ for all $x \in E$, we say that E is <u>bounded below</u>, and call y a <u>lower bound</u> of E.

Definition 3.3. Let S be a compact Hausdorff space irreducibly connected between two points. Let E be a subset of S.

Suppose y has the following properties (a) y is a lower bound of E. (b) If $x > y$, then x is not a lower bound of E. Then y is called the <u>greatest lower bound</u> of E [clearly from (b), there is at most one such y.] We shall use the abbreviation "inf." or "g.l.b." for "<u>greatest lower bound</u>".

<u>Lemma</u> 3.4. <u>Let S be a compact Hausdorff topological space which is irreducibly connected between two points a and b. Then every non-void subset E of S has a greatest lower bound.</u>

<u>Proof</u>. If E consists of a single element the result is immediate. Thus we may take E to be non-degenerate. Let B be the following set of elements: $\beta \in B$ iff there is an $x \in E$ such that $\beta > x$. Let $A = S - B$. Clearly, no member of B is a lower bound of E, and every member of A is a lower bound of E. To prove the existence of the inf., it suffices to prove that A has a greatest member. We now verify that A and B satisfy the hypothesis of Lemma 3.3.

Evidently (1) and (2) hold. Since E is non-degenerate, $B \neq \square$. Since E is bounded below, there exists a $y \in S$ such that $y \leq x$ for all $x \in E$. Thus $y \in A$, and $A \neq \square$. Hence (3) holds. If $\beta \in B$, there is an $x \in E$ such that $\beta > x$. If $\alpha \in A$, $\alpha \leq x$. Thus, $\alpha < \beta$ for all $\alpha \in A$ and $\beta \in B$ and (4) is satisfied.

Hence, by Corollary 3.1, either A contains a largest element or B contains a smallest element. We prove the second alternative cannot hold. Let $\beta \in B$. Then there is $x \in E$ such that $\beta > x$. Choose β' such that $\beta > \beta' > x$. Since $\beta' > x$, $\beta' \in B$ so that β is not the smallest member of B. This completes the proof.

Definition 3.4. A groupoid G with a zero is said to satisfy the cancellation law iff $xc = yc$ or $cx = cy$ implies that $x = y$ for all x, y, $c \in G$ with $c \neq 0$.

<u>Theorem</u> 3.1. <u>Let</u> G <u>be a compact groupoid with a zero and a unit such that</u> G <u>is irreducibly connected between</u> 0 <u>and</u> u. <u>Suppose</u> G <u>satisfies the cancellation law</u>. <u>Then</u> $x < y$ <u>implies</u> $xc < yc$ <u>and</u> $cx < cy$ <u>for all</u> x, y, $c \in G$ <u>with</u> $c \neq 0$.

<u>Proof</u>. Let $x, y \in G$ such that $x < y$. Let $A = \{x_\alpha : xc < yc \text{ for all } c \in [x_\alpha, u]\}$. Then $A \neq \square$ since $u \in A$. Let $k = \inf. \{x_\alpha \mid x_\alpha \in A\}$. Clearly, k exists by Lemma 3.4. We suppose $k > 0$. We next show that $xk \leq yk$. Suppose $xk > yk$. Then there exist disjoint open neighborhoods $V(xk)$ and $V(yk)$ of xk and yk respectively such that $\alpha \in V(xk)$ and $\gamma \in V(yk)$ imply $\alpha > \gamma$. There is a neighborhood $V(k)$ of k such that $xV(k) \subset V(xk)$ and $yV(k) \subset V(yk)$. But there is an $x_\alpha \in V(k)$. Hence, $xx_\alpha > yx_\alpha$ and we have a contradiction. If $xk = yk$, then $x = y$ by the cancellation law and we have a contradiction. Thus $xk < yk$. Hence there exist disjoint neighborhoods $V(xk)$ and $V(yk)$ of xk and yk respectively such that $\alpha \in V(xk)$ and $\gamma \in V(yk)$ imply $\alpha < \gamma$. But, there is a neighborhood $V(k)$ of k such that $xV(k) \subset V(xk)$ and $yV(k) \subset V(yk)$. We

may take $V(k)$ to be an open interval about k. Since there is an $x_\alpha \in V(k)$, $xc < yc$ for all $c \in [x_\alpha, u]$. But, $xc < yc$ for all $c \in V(k)$. There exists a $t \in V(k)$ such that $t < k$. Hence $xc < yc$ for all $c \in [t,u]$ and $t \in A$. This contradicts the definition of k. Hence, $k = 0$. Let $c \in G$ and suppose $c \neq 0$. Then there is an $x_\alpha \in [0, c)$ and $xc < yc$. A similar argument yields $cx < cy$ for $c \neq 0$.

Lemma 3.5. *Let G be a groupoid with a zero and a unit. If G satisfies the cancellation law, G contains no idempotents except 0 and u.*

Proof. Let f be an idempotent of G and suppose $f \neq 0$. Then $f = ff = fu$, and hence $f = u$ by the cancellation law.

Corrollary 3.2. *If G is a compact groupoid with a zero and a unit which is irreducibly connected between 0 and u and which satisfies the* cancellation law, then

(1) $xy \leq \min(x, y)$

(2) *if* $x \leq y$ *and* $w \leq v$, *then* $xw \leq yv$

(3) *if* x *has property* $\mathcal{A}$, $\{x^n : n \in I\}$ *converges to an idempotent. If* $x \neq u$, *then* $\{x^n : n \in I\}$ *converges to* 0.

Proof. (1). $x \leq u$ implies $xy \leq y$ and $y \leq u$ implies $xy \leq x$ by Theorem 3.1. (2) $x \leq y$ implies $xw \leq yw$ and $w \leq v$ implies $yw \leq yv$ by Theorem 3.1. Hence $xw \leq yv$. (3) Since x has property $\mathcal{A}$, $u \geq x \geq x^2 \geq x^3 \ldots$ by Theorem 3.1. By Theorem 2.4, $\{x^n : n \in I\}$ must cluster at an idempotent f. But, it follows from Lemma 3.4 that the monotone decreasing sequence $\{x^n : n \in I\}$ has a limit.

Hence $\lim_n x^n = f$. If $x \neq u$, then $\{x^n : n \in I\}$ is bounded away from u , and $\lim_n x^n \neq u$. Thus by Lemma 3.5, $\lim_n x^n = 0$.

Lemma 3.6. *Let G be a compact groupoid with a zero and a unit which is irreducibly connected between 0 and u and which satisfies the cancellation law. Then $Gx = [0, x] = xG$ for all x in G .*

Proof. Let $x \in G$. Since Gx is a connected set containing 0 and x , $Gx \supset [0, x]$ by Lemma 3.2. If $z \in Gx$, $z = ax$ for some $a \in G$ and $z \leq x$ by Corollary 3.2. Hence, $z \in [0, x]$ and $Gx \subset [0, x]$. Thus $Gx = [0, x]$. By a similar argument, $xG = [0, x]$.

Lemma 3.7. *Let G be a compact groupoid with a zero and a unit which is irreducibly connected between 0 and u . Suppose G satisfies the cancellation law and every element of G has property $\mathcal{a}$. Then G is an arc and G is an abelian topological semi-group.*

Proof. Let Δ^n be the diagonal of the cartesian product of G , n times. For each n define $f_n : \Delta^n \longrightarrow G$ by $f_n(y) = x^n$, where x is the projection of y into G . (We note that all powers of x are unique for all $x \in G$ by virtue of Lemma 2.3). Since f_n is continuous, $f_n(\Delta^n) \subset G$ is a connected set containing 0 and u . Since G is irreducibly connected between 0 and u , $f_n(\Delta^n) = G$ This implies f_n is onto and, in particular, that any $x \in G$ has an n^{th} root.

We now assert that if $x \neq 0$, x has a unique square root. Suppose $a^2 = b^2 = x \neq 0$. Suppose $a \leq b$. Then $a^2 \leq ab \leq b^2 = a^2$ by Theorem 3.1. Hence $ab = aa$. Thus, since $a \neq 0$, $a = b$ by the cancellation law. By induction, x has a unique 2^{m}th root.

Let $x, y \in G$. Suppose $y < x$. Then $y^2 < x^2$ by Theorem 3.1. It follows $y^{2^m} < x^{2^m}$ for all positive integers m . We call this result (1).

Let us define for $x \in G$, $x \neq 0$

$$(x^{p/2^q}) = (x^{1/2^q})^p$$

where p and q are positive integers. It is immediate from the fact that the 2^q th roots are unique, that x^r is well defined for any positive dyadic rational r, regardless of the representation of r. By the same argument, straight forward verification establishes that we can write

$$x^r \cdot x^s = x^{r+s} \quad . \tag{2}$$

We claim further that if r and s are dyadic rationals such that $r < s$, then $x^r \geqslant x^s$. There exists a dyadic rational t such that $r + t = s$. By (2), $x^s = x^{r+t} = x^r \cdot x^t$. By Corollary 3.2, $x^s = x^r x^t \leqslant x^r$. Therefore, if $\{r_n\}$ is a monotone increasing sequence of dyadic rationals, $\{x^{r_n}\}$ is a monotone decreasing sequence in G.

For $x \in G$, $x \neq 0, u$, let $D = \{x^r : r$ is a positive dyadic rational$\}$. By (2), D is an abelian submob of G. We claim $\overline{D} = G$.

Let $V(0)$ be any neighborhood of zero. Since $x \neq u$, $x^n \longrightarrow 0$ by Corollary 3.2. Thus, $V(0) \cap D \neq \square$. Let $V(u)$ be any basis element containing u. Let $y \in V(u)$ with $y < u$. Clearly $y^n \longrightarrow 0$. Thus there exists a positive integer n such that $y^{2^n} < x$. Hence, by (1) and the linearity of the ordering $y < x^{1/2^n}$ and $V(u) \cap D \neq \square$.

Let t be an arbitrary element of G, $t \neq 0$ and $t \neq u$ and let $B = \{s : a < s < b\}$ be an arbitrary basis element in the order topology containing t. Without any loss of generality, we may assume

$0 < a < b < u$. Let

$$R' = \{r : x^r \geq b\ ,\ r \text{ a positive dyadic rational}\}$$
$$R'' = \{r : x^r \leq a\ ,\ r \text{ a positive dyadic rational}\}$$

Clearly, R' and R'' are non-empty. Since $r < s$ implies $x^r \geq x^s$, R' and R'' have upper and lower bounds respectively. Let r' be the l.u.b. of R' and r'' be the g.l.b. of R'' . Since $\{x^r\}$ is a monotone decreasing sequence as r increases, $r' \leq r''$. If $r' < r''$, D meets B . Therefore, we assume $r' = r''$. Let $\{p_n\}$ be a monotone increasing sequence of dyadic rationals of R' converging to r' . Then $\{x^{p_n}\}$ is a monotone decreasing sequence such that $\{x^{p_n}\} \geq b$ for all n. By Lemma 3.4, $\{x^{p_n}\}$ has a g.l.b. $c \geq b$. Hence $\{x^{p_n}\}$ converges to c . Let W be an open set containing c so that $W \cap [0, a] = \Box$. By the continuity of multiplication, there exist open sets U and V with $u \in U$ and $c \in V$ such that $UV \subset W$. Since $D \cap U \neq \Box$, $x^r \in U$ for some r . There exists a p_k in the sequence $\{p_n\}$ such that $x^{p_k} \in V$ and $r + p_k > r' = r''$. This implies $x^{r+p_k} \in [0, a]$. But, $x^{r+p_k} = x^r \cdot x^{p_k} \in UV \subset W$ and W is disjoint from $[0, a]$. From this contradiction it follows that $\bar{D} = G$. Thus G is a mob by Lemma 2.1. Clearly G is abelian. Since D is a countable dense set and since G is compact and irreducibly connected between two points, G is homeomorphic to I , the unit interval of real numbers [21].

Example 3.1. An example of a finite groupoid with a zero and a unit which obeys the cancellation law and in which every element has property $\mathcal{a}$, but which is non-associative and non-abelian is given by $G = \{1,2,3,4,5\}$

with following multiplication table.

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	1	5	3	4
3	0	3	4	1	5	2
4	0	4	5	2	1	3
5	0	5	3	4	2	1

We see

$$(23)4 \neq 2(34) \quad \text{and} \quad 2 \cdot 3 \neq 3 \cdot 2 .$$

Definition 3.5. An algebraic nilpotent of a groupoid with a zero is an element x such that x^n is unique for some n and for this n, $x^n = 0$.

Definition 3.6. An algebraic nilpotent of a semi-group is an element x such that $x^n = 0$ for some n.

We next state a result due to Faucett [3]..

Lemma 3.8. *Let S be a compact semi-group with a zero and a unit and no other idempotents. Suppose S is irreducibly connected between 0 and u. Assume further that S contains no non-zero algebraic nilpotents. Then there exists a function F from S onto I, the unit interval $[0, 1]$ of real numbers with the usual topology and the usual multiplication, that is an isomorphism as well as an order preserving homeomorphism.*

Lemma 3.9. *Let G be a groupoid with a zero such that every element of G has property $\mathcal{A}$ and G satisfies the cancellation law. Then G contains no non-zero algebraic nilpotents.*

Proof. Suppose $x^n = 0$. Then $x^{n-1} \circ x = 0 \circ x$. Thus, $x = 0$ or $x^{n-1} = 0$. Similarly, $x = 0$ or $x^{n-2} = 0$. Proceeding in this way, we see that $x = 0$.

Theorem 3.2. Let G be a compact groupoid with a zero and a unit which is irreducibly connected between 0 and u . Assume every element of G has property $\mathcal{a}$ and that G satisfies the cancellation law. Then there exists a function f from G onto I , the unit interval [0, 1] of real numbers under the usual topology and the usual multiplication, which is an isomorphism as well as an order preserving homeomorphism.

Proof. This result follows immediately from Lemmas 3.5, 3.7, 3.8, and 3.9.

Corollary 3.3. Let G be a compact groupoid with a zero and another idempotent f which is center associative. Assume G is irreducibly connected between 0 and f , G satisfies the cancellation law, and every element of G has property $\mathcal{a}$. Then there exists a function f from G onto I , the unit interval [0, 1] of real numbers under the usual topology and the usual multiplication, which is an isomorphism as well as an order preserving homeomorphism.

Proof. Since fG is a connected set containing 0 and f , fG = G . Similarly, Gf = G . Since f is a center associative idempotent f acts as a right unit on Gf = G and a left unit on fG = G . Hence f is a two sided unit on G and the result follows from Theorem 3.2.

Remark. We note that any continuous multiplication on [0, 1] with the usual topology is the usual multiplication for real numbers if the multiplication obeys the cancellation law, 0 acts as a zero, 1 acts as a unit, and every element of [0, 1] has property $\mathcal{a}$.

Example 3.2 An example of a continuous multiplication on $[0, 1]$ with the usual topology which is non-associative and non-abelian, which has 0 as a zero, which has 1 as a right unit, and which obeys the cancellation law. We define: $a \circ b = ab^2$ where the last multiplication is that of real numbers.

We next consider the associative-like properties possessed by compact groupoids with a zero and a unit which are irreducibly connected between 0 and u and which obey the cancellation law.

Lemma 3.10. Let G be a compact groupoid with a zero and a unit which is irreducibly connected between 0 and u and which satisfies the cancellation law. Then $(Ga)b = G(ab)$ and $a(bG) = (ab)G$ for all a and b in G.

Proof. $[0, ab] \subset [0, a]b$ by Lemma 3.2. Thus, $(Ga)b = [0, a]b \supset [0, ab] = G(ab)$ by virtue of Lemma 3.6. We will next show that $[0, a]b \subset [0, ab]$. Let $z \in [0, a]b$. Then $z = xb$ where $0 \leq x \leq a$. Hence, $0 \leq z \leq ab$ by Corollary 3.2 and $z \in [0, ab]$. Now, $(Ga)b = [0, a]b \subset [0, ab] = G(ab)$ by Lemma 3.6 and $(Ga)b = G(ab)$. Similarly, $a(bG) = (ab)G$.

Lemma 3.11. Let G be a compact groupoid with a zero and a unit which is irreducibly connected between 0 and u and which satisfies the cancellation law. Then $Gx = xG = [0, x]$ is a two sided ideal of G.

Proof. $a(Gx) = a[0, x] = [0, ax] \subset [0, x] = Gx$ by Corollary 3.2, Lemma 3.2, and Lemma 3.6. Similarly $(Gx)a = [0, x]a = [0, xa] \subset [0, x] = Gx$. Thus Gx is a two sided ideal. Also, xG is a two sided ideal.

Lemma 3.12. If G is a compact groupoid with a zero and a unit which is irreducibly connected between 0 and u and which obeys the cancellation law, $(Gx)G = G(xG)$ and $(Gx)G = G(xG)$ is a two-sided ideal for all $x \in G$.

Proof. We first note that $[0, x][0,u] = [0,x]$ for all $x \in G$. Let $z \in [0, x][0, u]$. Then $z = yt$ where $0 \leq y \leq x$ and $0 \leq t \leq u$. Thus, $0 \leq yt \leq xt \leq x$ by Corollary 3.2 and $z \in [0, x]$. If $z \in [0, x]$, $z = zu$ and $z \in [0, x][0, u]$. Hence, $(Gx)G = [0, x][0, u] = [0, x]$ by virtue of Lemma 3.6. By the above method, it is easily shown that $G(xG) = [0, x]$. Thus, $G(xG) = (Gx)G = [0, x]$.

Lemma 3.13. If G is a compact groupoid with a zero and a unit which is irreducibly connected between 0 and u and which obeys the cancellation law, $a(Gb) = (aG)b$ for all a and b in G .

Proof. $a(Gb) = a[0, b] = [0, ab] = [0, a]b = (aG)b$ by Corollary 3.2, Lemma 3.2, and Lemma 3.6.

Lemma 3.14. If G is a compact groupoid with a zero and a unit which is irreducibly connected between 0 and u and which satisfies the cancellation law, then $(GV)G = G(VG)$ and $(GV)G = G(VG)$ is a two sided ideal of G for all subsets V of G .

Proof. If $z \in (GV)G$, $z \in (Gv)G$ for some $v \in V$. Hence, $z \in G(vG) \subset G(VG)$ by Lemma 3.12 and $(GV)G \subset G(VG)$. Similarly $G(VG) \subset (GV)G$. If $z \in GVG$, $z \in GvG$ for some v V . Hence, $za \in (GvG)a \subset GvG \subset GVG$ for all a in G by Lemma 3.12. Similarly, $az \in GVG$ for all a in G . Thus $GVG = (GV)G = G(VG)$ is a two sided ideal of G .

Lemma 3.15. If G is a compact groupoid with a zero and a unit which is irreducibly connected between 0 and u and which satisfies the cancellation law, then GV and VG are two sided ideals of G for all subsets V of G .

Proof. If $z \in GV$, $z \in Gv$ for some $v \in V$. Hence $az \in a(Gv) \subset Gv \subset GV$ for all a in G by Lemma 3.11. Similarly, $za \in GV$ for all $a \in G$ and GV is a two-sided ideal of G . By the same method, VG is a two-sided ideal of G .

We next apply the above "associative-like" properties to extend some of the results given by Koch and Wallace [12] and Numakara [16] for mobs to compact groupoids with a zero and a unit which are irreducibly connected between 0 and u and which obey the cancellation law.

Definition 3.7. If $A \subset G$, a groupoid, and A includes an ideal of G , let $J(A)$ be the union of all ideals contained in A . Thus $J(A)$ is the largest ideal contained in A . If A includes no ideal of G , $J(A) = \square$, the null set.

Theorem 3.3. Let G be a compact groupoid with a zero and a unit which is irreducibly connected between 0 and u and which satisfies the cancellation law. Then, if A is open, $J(A)$ is open.

Proof. If $J(A) = \square$, we are done. Suppose $J(A) \neq \square$. Let $x \in J(A)$. Then $x \cup xG \cup Gx \cup GxG \subset J(A) \subset A$ by Lemma 3.12. Since A is open and G is compact, we apply a lemma of Wallace [18, Lemma 1] and Lemma 3.14 to produce an open set V about x such that $V \cup VG \cup GV \cup GVG \subset A$. By Lemma 3.14 and Lemma 3.15, this set is an ideal of G and hence is contained in $J(A)$. Therefore $x \in V \subset J(A)$ and $J(A)$ is open.

Theorem 3.4. Let G be a compact groupoid with a zero and a unit which is irreducibly connected between 0 and u and which satisfies the cancellation law. Then $G \setminus u = [0, u)$ is the maximal proper ideal of G .

<u>It contains every proper ideal of</u> G .

<u>Proof.</u> If $a \neq 0$, $a[0, u) \subset [0, u)$ by Theorem 3.1. If $a = 0$, the result is trivial. Similarly, $[0, u)a \subset [0, u)$. Thus, $[0, u)$ is a two sided ideal of G . If A is any proper ideal of G , $u \bar{\in} A$. Thus, $A \subset G \setminus u$.

<u>Corollary 3.3.</u> <u>Let</u> G <u>be a compact groupoid with a zero and a unit which is irreducibly connected between</u> 0 <u>and</u> u <u>and which obeys the cancellation law. If</u> $H(u)$ <u>is any subgroup of</u> G <u>containing</u> u , <u>then</u> $H(u) = u$.

Proof. Clearly, $u \in H(u)$. If $x \in H(u)$, there exists an $\bar{x} \in H(u)$ such that $x\bar{x} = u$. If $x \in G \setminus u$, $x\bar{x} \in G \setminus u$ since $G \setminus u$ is an ideal by Theorem 3.4. Thus, $H(u) = u$.

CHAPTER IV

APPLICATIONS

Monothetic Groupoids

Definition 4.1. A groupoid G is said to be cyclic iff there exists an $a \in G$ such that every element of G can be expressed as a finite "word" in a ; and a is called a generator of G [2].

Definition 4.2. A topological groupoid G is monothetic iff there exists $a \in G$ such that the collection of all finite "words" in a is dense in G. The element a is called a generator of G.

Theorem 4.1. A compact topological groupoid G contains a compact monothetic subgroupoid A.

Proof. Let A be the collection of all finite "words" in $a \in G$. Since A is a groupoid, $\overline{A}$ is a groupoid, and A is dense in $\overline{A}$.

Definition 4.3. Let G be a groupoid and a an element of G. Let $\mathcal{a}_n = \{$collection of all n letter words in the letter $a\}$. There are $N(n) = \frac{(2n-2)!}{n!(n-1)!}$ such words [8]. Denote these in any way by $a_{n_1}, a_{n_2}, \ldots, a_{n_{N(n)}}$. Thus we obtain a sequence $\{a_{1_{N(1)}}, a_{2_{N(2)}}, \ldots, a_{n_1}, a_{n_2}, \ldots, a_{n_{N(n)}}, \ldots\}$. Relabel this sequence in such a manner that if $b_1 = a_{1_{N(1)}}$ and if $b_m = a_{n_{N(n)}}$ then the terms $b_{m+1}, \ldots, b_{m+N(n+1)}$ are the terms $a_{(n+1)_1}, \ldots, a_{(n+1)_{N(n+1)}}$. The theorems in this section are independent of the way we assign $\mathcal{a}_{n_1}, a_{n_2}, \ldots, a_{n_{N(n)}}$, to the elements of $\mathcal{a}_n$, i.e., the theorems are for all possible sequences $\{b_n : n \in I\}$.

Theorem 4.2. Let G be a monothetic groupoid with generator a. Then the net $\{b_n : n \in I\}$ clusters at each idempotent of G.

Proof. Suppose e is an idempotent of G and $e = b_n$ for some positive integer n. From this it is clear, if we are given any $k \in I$, there exists $m > k$ such that $b_m = e$, i.e. $\{b_n : n \in I\}$ clusters at e. Next, we suppose $e \neq b_i$ for all $i \in I$. Then, given any $p \in I$ and any open set $\mathcal{O}$ about e, there exists $k \in I$ such that $k > p$ and $b_k \in \mathcal{O}$. Otherwise, there exists an open set $\mathcal{O}$ about e such that for any positive integer k, $b_k \bar{\in} \mathcal{O}$. But this contradicts the fact that A is dense in G. Thus, $\{b_n : n \in I\}$ clusters at e.

Theorem 4.3. A necessary and sufficient condition for a monothetic groupoid G to be a monothetic semi-group [11] is that its generator have property $\mathcal{a}$.

Proof. If the generator a of G has property $\mathcal{a}$, all the powers of a are unique by Lemma 2.3. Hence $\overline{\{a^n : n \in I\}} = G$. Since $\{a^n : n \in I\}$ is a semi-group, G is a semi-group by Lemma 2.1. Clearly a monothetic topological semi-group has a generator with property $\mathcal{a}$.

Theorem 4.4. Let G be a monothetic groupoid with a unit and suppose that $G = P \cup Q$ where P and Q are disjoint, closed, and non-empty. Then P and Q each contain infinitely many finite words in any generator a of G.

Proof. Let a be a generator of G. Denote the unit of G by 1 and assume $1 \in P$. Since P and Q are open, each contains elements of $\{b_n : n \in I\}$. To establish the conclusion for P, suppose there is a largest integer k such that $b_k \in P$. Then the net $\{b_{k+i}\}_{i=1}^{\infty}$ is a net in Q. But, $\{b_{k+i}\}_{i=1}^{\infty}$ clusters at 1 by

Theorem 4.2. Thus $1 \in Q$ since Q is closed and we have a contradiction. To establish the conclusion for Q, suppose there exists a largest integer k such that $b_k \in Q$. Then the net $\{b_{k+i}\}_{i=1}^{\infty}$ is a net in P and clusters at 1. Hence by continuity $\{b_k b_{k+i}\}_{i=1}^{\infty}$ clusters at b_k, but the latter net is in P. Thus $b_k \in P$ since P is closed and we have a contradiction.

Theorem 4.5. *If G is a compact monothetic groupoid $\bigcap_{n=1}^{\infty} \overline{\{b_i : i \geq n\}}$ is a closed ideal of G.*

Proof. Let $a \in G$ and $b \in \bigcap_{n=1}^{\infty} \overline{\{b_i : i \geq n\}}$. Suppose $V(ab)$ is an arbitrary neighborhood of ab. Then there exist neighborhoods $V(a)$ and $V(b)$ of a and b respectively such that $V(a)V(b) \subset V(ab)$. But there exist $b_j \in V(a)$ for some $j \in I$ and $b_{n+n_\gamma} \in V(b)$ for all $n \in I$ where n_γ is a positive integer. Hence $b_j b_{n+n_\gamma} = b_{n+n_k} \in V(ab)$ for all positive integers n and $ab \in \bigcap_{n=1}^{\infty} \overline{\{b_i : i \geq n\}}$. Clearly, $\bigcap_{n=1}^{\infty} \overline{\{b_i : i \geq n\}} \neq \Box$. Thus $\bigcap_{n=1}^{\infty} \overline{\{b_i : i \geq n\}}$ is a left ideal of G. Similarly, it is right ideal of G.

Definition 4.4. If G is a groupoid and B is a subset of G, B^n denotes the totality of n letter words with letters in B.

Lemma 4.1. *Let G be a compact groupoid with a unit and $a \in G$. Then $\bigcap_{n=1}^{\infty} \overline{\{b_i : i \geq n\}} = \bigcap_{n=1}^{\infty} G^n$ if and only if $A \subset G$, $A = \overline{A}$, and $AA = A$ implies $A \subset \bigcap_{n=1}^{\infty} \overline{\{b_i : i \geq n\}}$.*

Proof. We first assume $\bigcap_{n=1}^{\infty} \overline{\{b_i : i \geq n\}} = \bigcap_{n=1}^{\infty} G^n$. Suppose $A \subset G$ and $AA = A$. Then $A^n = A$ for all positive integers n. Thus, $A \subset \bigcap_{n=1}^{\infty} G^n = \bigcap_{n=1}^{\infty} \overline{\{b_i : i \geq n\}}$. We next assume $A \subset G$, $A = \overline{A}$, and $AA = A$ implies $A \subset \bigcap_{n=1}^{\infty} \overline{\{b_i : i \geq n\}}$. Clearly, $\bigcap_{n=1}^{\infty} \{b_i : i \geq n\} \subset \bigcap_{n=1}^{\infty} G^n$. Hence it is easily seen that $\bigcap_{n=1}^{\infty} \overline{\{b_i : i \geq n\}} \subset \bigcap_{n=1}^{\infty} \overline{G^n} = \bigcap_{n=1}^{\infty} G^n$. Let

$A = \bigcap_{n=1}^{\infty} G^n$. Since A is a two sided ideal, $AA \subset A$. Suppose $AA \not\supset A$. Then there exists $x \in A$ such that $x \notin AA$. Since G is compact and A is closed, there exists an open set $V \supset A$ such that $x \notin VV$ [18]. By use of an elementary net argument, it may easily be shown there exists an integer m such that $G^m \subset V$. Hence $x \notin G^m G^m \supset G^{2m}$. Hence, we have a contradiction and $AA = A$. Thus, $A = \bigcap_{n=1}^{\infty} G^n \subset \bigcap_{n=1}^{\infty} \overline{\{b_i : i \geqslant n\}}$. Hence $\bigcap_{n=1}^{\infty} G^n = \bigcap_{n=1}^{\infty} \overline{\{b_i : i \geqslant n\}}$.

<u>Theorem</u> 4.6. <u>If</u> G <u>is a compact monothetic groupoid, then</u> $A \subset G$, $A = \bar{A}$, <u>and</u> $AA = A$ implies $A \subset \bigcap_{n=1}^{\infty} \overline{\{b_i : i \geqslant n\}}$.

Proof. Let a be a generator of G . We will first show that $G^n \subset \overline{\{b_i : i \geqslant n\}}$. Let $x \in G^n$ and suppose $x \notin \overline{\{b_i : i \geqslant n\}}$. Then x is not a cluster point of $\{b_n\}$. Therefore there exists $k \in I$ such that $x = b_k$ where $k < n$ since $\overline{\{b_n : n \in I\}} = G$. Since $x \in G^n$, there exist elements $c_1, c_2, \ldots, c_n$ in G such that $x = b_k$ is a word with letters $c_1, c_2, \ldots, c_n$. We denote b_k by $W(c_1, c_2, \ldots, c_n)$. No $c_i \in \overline{\{b_i i \geqslant n\}}$. Suppose $c_i \in \overline{\{b_i i \geqslant n\}}$. Let $V(W(c_1, c_2, \ldots, c_n))$ be an arbitrary neighborhood of $W(c_1, c_2, \ldots, c_n)$. Then there exists neighborhoods $V(c_1)$, $V(c_2), \ldots,$ and $V(c_n)$ of c_1 , $c_2, \ldots,$ and c_n respectively such that $W(V(c_1), \ldots, V(c_n)) \subset V(W(c_1, c_2, \ldots, c_n))$. But there exists $b_{r_1} \in V(c_1)$, $b_{r_2} \in V(c_2)$, $\ldots, b_{n+n_r} \in V(c_i)$, $\ldots, b_{r_n} \in V(c_n)$. Now, $W(b_{r_1}, b_{r_2}, \ldots, b_{n+n_r}, \ldots, b_{r_n}) = b_t$ where $t \geqslant n$. Hence $x = W(c_1, c_2, \ldots, c_n)) \in \overline{\{b_i : i \geqslant n\}}$ and we have a contradiction. Thus, as asserted above no $c_i \in \overline{\{b_i : i \geqslant n\}}$. Thus, each of the elements $c_1, c_2, \ldots, c_n$ is a b_i where $i < n$. Therefore $b_k = b_p$

where $p \geqslant n > k$. Thus, $b_k \in \overline{\{b_i : i \geqslant n\}}$ and we have a contradiction. Hence $G^n \subset \overline{\{b_i : i \geqslant n\}}$ for all n and $\bigcap_{n=1}^{\infty} G^n \subset \bigcap_{n=1}^{\infty} \overline{\{b_i : i \geqslant n\}}$. As above, $\bigcap_{n=1}^{\infty} \overline{\{b_i : i \geqslant n\}} \subset \bigcap_{n=1}^{\infty} G^n$. Thus, $\bigcap_{n=1}^{\infty} G^n = \bigcap_{n=1}^{\infty} \overline{\{b_i : i \geqslant n\}}$. Hence $A \subset G$, $A = \overline{A}$, and $AA = A$ implies $A \subset \bigcap_{n=1}^{\infty} \overline{\{b_i : \geqslant n\}}$ by Lemma 4.1.

If G is a monothetic semi-group, $\bigcap_{n=1}^{\infty} \overline{\{b_i : i \geqslant n\}}$ is the minimal two sided ideal of G [19]. In this special case, Theorems 4.2, 4.4, and 4.6 have been proved by Koch ([10] and [11, Section 3])

Example 4.1. An example of a finite cyclic groupoid which is not a cyclis semi-group is $G = 1, 2, 3$ with the following multiplication table.

	1	2	3
1	2	3	x
2	3	x	1
3	x	x	1

where x may be 1, 2, or 3. Clearly, $(12)3 \neq 1(23)$ and 1 is a generator of G.

Example 4.2. An example of an infinite cyclic groupoid which is not a cyclic semi-group. Consider the set $a_1, a_2, \ldots, a_3, \ldots$ where $a_i = a_j$ iff $i = j$. Define the following multiplication: $a_i a_j = a_1$ if $i \neq j$ and $a_i a_i = a_{i+1}$. Clearly $(a_1 a_2) a_3 \neq a_1(a_2 a_3)$ and a_1 is a generator.

If we use the discrete topology in Example 4.1 and Example 4.2 we obtain monothetic groupoids which are not monothetic semi-groups.

Example 4.3. Consider the interval $[0, 1]$ of real numbers under the usual topology and the following multiplication: $a \circ b = ab^2$ where the multiplication on the right is the usual real multiplication. Let $a = \frac{1}{2}$. Then $\overline{\{b_n : n \in I\}}$ is a compact monothetic groupoid with generator $\frac{1}{2}$ which is not a monothetic semi-group and which has zero as its only idempotent. We note that if $A \subset G = \overline{\{b_n : n \in I\}}$, $A = \bar{A}$, and $AA = A$, then $A = 0 \subset \bigcap_{n=1}^{\infty} \overline{\{b_i : i \geqslant n\}}$.

Example 4.4. Consider the integers with the usual topology under subtraction. They form a monothetic topological groupoid which is not a topological semi-group. One is a generator of this groupoid.

Topological Groupoids With A Zero

Definition 4.5. Let G be a topological groupoid with a zero and a be an element of G such that a^n is unique for all n. If $a^n \longrightarrow 0$, i.e., if for every neighborhood U of 0, there exists a positive integer $m = m(U, a)$ such that $a^n \in U$ for all $n \geqslant m$, then a is termed a <u>nilpotent element</u> or in short a <u>nilpotent</u>.

Definition 4.6. <u>A nil left (right, two sided) ideal</u> of a topological groupoid G with a zero is a left (right, two sided) ideal of G which consists entirely of nilpotent elements. A <u>nil semi-group</u> (sub-groupoid) of G is a semi-group (sub-groupoid) of G consisting entirely of nilpotent elements.

<u>Lemma</u> 4.2. <u>Let</u> $\{A_\lambda : \lambda \in \Lambda\}$ <u>be a family of right (left, two-sided) nil ideals of a topological groupoid with a zero. Then</u> $A = \bigcup_{\lambda \in \Lambda} A_\lambda$ <u>is also a right (left, two-sided) nil ideal of</u> G.

<u>Theorem</u> 4.7. <u>If</u> G <u>is an elementwise bicompact groupoid with a zero, every right (left, two-sided) ideal</u> A <u>of</u> G <u>is either a nil ideal</u>

<u>or contains non-zero idempotents provided every element of</u> A <u>has property</u> $\mathcal{a}$.

<u>Proof</u>. Let A be a non-nil right ideal of G . Then there is an element $a \neq 0$ in A such that $a^n \not\to 0$ since for all $a \in A$, the powers of a are unique by Lemma 2.3. Now, if we consider subsets $B = \{a^n : n \in I\}$, $B_\gamma = \{a^i : i \geq \gamma\}$ ($\gamma = 1, 2, 3, \ldots$), and $D = \bigcap_{\gamma=1} \overline{B_\gamma}$, D is a subgroup of G and $BD \subseteq D$ by Theorem 2.4. Let e be the identity of D . If $e = 0$, $D = (0)$ and $a^n \to 0$. But, this contradicts the assumption that $a^n \not\to 0$. Thus, e is a non-zero idempotent. Since $BD \subseteq D$, $ae \in D$ and we denote by $(ae)^{-1}$ the inverse of ae in D . Since A is a right ideal, $e = (ae)(ae)^{-1} \in A$. Hence A contains a non-zero idempotent e . It is the same with the case of the left ideal.

<u>Theorem</u> 4.8. <u>A closed subgroupoid</u> G_1 <u>of an elementwise bicompact groupoid with a zero is either a nil subgroupoid or contains non-zero idempotents provided each element of</u> G_1 <u>has property</u> $\mathcal{a}$.

<u>Proof</u>. The existence of the non-zero idempotent e is established as in the proof of Theorem 4.7. One then notes that $e \subset D \subset \overline{G}_1 = G_1$.

Definition 4.7. A topological groupoid with a zero is said to be an N-groupoid if its nilpotent elements form an open set.

<u>Lemma</u> 4.3. <u>Let</u> G <u>be a topological groupoid with a zero and let</u> a <u>be an element of</u> G <u>with property</u> $\mathcal{a}$. <u>If</u> a^n <u>is a nilpotent element for some positive integer</u> n , <u>then</u> a <u>itself is a nilpotent element</u>.

<u>Proof</u>. All the powers of a are unique by Lemma 2.3. Let $b = a^n$ and let U be an arbitrary neighborhood of 0 . Then there exist neighborhoods W_i $(i = 1, 2, \ldots, n-1)$ of zero such that $W_i a^i \subset U$

$(i = 1, 2, \ldots, n-1)$. Now, since $b^\gamma \longrightarrow 0$, there exist integers M_i $(i = 1, 2, \ldots, n-1)$ such that $\gamma \geqslant M_i$ implies $b^\gamma \in W_i$ and there exists a positive integer M_0 such that $\gamma \geqslant M_0$ implies $b^\gamma \in U$. Let $M^* = \max(M_0, M_1, \ldots, M_{n-1})$ and let $M = n(M^* + 1)$. If $u \geqslant M$, we write $u = kn + i$, $0 \leq i \leq n - 1$ [18]. Then $k \geqslant M^*$. If $i = 0$, we have $a^u = a^{kn} = b^k \in U$. If $i \neq 0$, we have $a^u = a^{kn+i} = a^{kn}a^i = b^k a^i \in W^i a^i \subset U$. Hence a is a nilpotent.

Definition 4.8. If G is a topological groupoid and $A \subset G$, we shall use the following notation: $\hat{A} = A$, $\widehat{A^2} = \hat{A}A$, $\widehat{A^3} = \widehat{A^2}A$, ..., $\widehat{A^n} = \widehat{A^{n-1}}A$, etc.

<u>Theorem</u> 4.9. <u>Let</u> G <u>be a topological groupoid with a</u> 0 <u>in which every element has property</u> $\mathcal{a}$. <u>If</u> G <u>has a neighborhood</u> V_0 <u>of</u> 0 <u>which consists entirely of nilpotents, then</u> G <u>is an</u> N-<u>groupoid</u>.

<u>Proof</u>. We denote the set of all nilpotent elements of G by N . If p is an element of N , there exists a positive integer n such that $p^n \in V_0$. Recalling the above convention, one can easily see that there is a neighborhood $U(p)$ of p such that $\widehat{U(p)^n} \subset V(p^n) \subset V_0$. If q is any element of $U(p)$, then $q^n = \widehat{q^n} \in V_0$ and q^n is a nilpotent. Thus, q is a nilpotent by Lemma 4.3 and $p \in U(p) \subset N$. Hence, G is an N-groupoid.

Definition 4.9. The join of all left nil ideals of a topological groupoid with a zero is called the <u>radical</u> of G .

Definition 4.10. An element b of a groupoid G is said to be right associative iff $(xy)b = x(yb)$ for all x and y in G [14].

<u>Theorem</u> 4.10. <u>The radical of a bicompact</u> N-<u>groupoid is open provided each nilpotent element is right associative</u>.

<u>Proof</u>. Let R be the radical of G , and let $a \in R$. Then for any $x_\lambda \in G$, $x_\lambda a \subset R \subset N$, where N is the set of all nilpotent

elements of G. The above relation is valid since R is a left nil ideal of G by virtue of Lemma 4.2. Since N is open by assumption, there exists a neighborhood $V(x_\lambda a)$ of $x_\lambda a$ such that $V(x_\lambda a) \subset N$ for each $x_\lambda \in G$. Hence, one can find neighborhoods $V(x_\lambda)$ of x_λ and $V_\lambda(a) \subset N$ of a such that $V(x_\lambda)V_\lambda(a) \subset V(x_\lambda a)$. Since G is bicompact and $\bigcup_{x_\lambda \in G} V(x_\lambda) = G$, one can find a finite subcollection $\{V(x_{\lambda i}) : i = 1, 2, \ldots, k\}$ of $\{V(x_\lambda) : x_\lambda \in G\}$ so that $\bigcup_{i=1}^{k} V(x_{\lambda i}) = G$. We denote by $V(a)$ a neighborhood of a such that $V(a) \subset \bigcap_{i=1}^{k} V_{\lambda i}(a)$ where $V_{\lambda i}(a)$ corresponds to $V(x_{\lambda i})$, $i = 1, 2, \ldots, k$. Thus $GV(a) \subset \bigcup_{i=1}^{k} V(x_{\lambda i})V(a) \subset \bigcup_{i=1}^{k} V(x_{\lambda i}a) \subset N$. If we put $L^* = GV(a) \cup V(a)$, then L^* is a left nil ideal since each nilpotent element is right associative. Thus, $a \in V(a) \subset L^* \subset R$ and R is open.

<u>Example</u> 4.5. Let $[0, 1]$ be the closed unit interval of real numbers under the usual topology. Define the following multiplication on $[0, 1]$: $a \circ b = ab^2$ for all $a, b \in [0, 1]$ where the multiplication on the right is the usual multiplication of real numbers. A topological groupoid G is formed in which 0 is the only nilpotent element. Hence, G is not an N-groupoid. But, if we consider $[0, 1]$ with the discrete topology and the above multiplication, we form a topological groupoid G' which is an N-groupoid.

We next consider briefly the concept of boundedness in topological groupoids with a zero.

Definition 4.11. A subset B of a topological groupoid with a zero is right bounded iff for any neighborhood U of 0 there exists a neighborhood V of 0 such that $VB \subset U$. Left boundedness is similarly defined, and a set is bounded iff it is right bounded and left bounded.

<u>Theorem 4.11. A bicompact subset of a topological groupoid with a zero is bounded.</u>

<u>Proof</u>. Let B be a bicompact subset of G. Let $U(0)$ be an arbitrary neighborhood of 0. Since $0x = 0$ for $x \in B$, there exist neighborhoods $V(x)$ of x and $W_x(0)$ of 0 such that $W_x(0)V(x) \subset U(0)$. Since B is bicompact and $\bigcup_{x \in B} V(x) \supset B$, there exists a finite subcollection $\{V(x_i): i = 1, 2, \ldots, k\}$ of $\{V(x): x \in B\}$ such that $\bigcup_{i=1}^{k} V(x_i) \supset B$. Let $W(0) = \bigcap_{i=1}^{k} W_{x_i}(0)$ where $W_{x_i}(0)$ corresponds to $V(x_i)$, $i = 1, 2, \ldots, k$. Hence $W(0)B \subset U(0)$ and B is right bounded. Similarly, B is left bounded.

We finally consider the special case of a bicompact groupoid with a zero and a unit which is irreducibly connected between 0 and u and which obeys the cancellation law.

<u>Theorem 4.12. Let G be a bicompact groupoid with a zero and a unit which is irreducibly connected between 0 and u and which obeys the cancellation law. Then</u>

(1) $[0, x)$ <u>is a nil ideal of</u> G <u>provided all</u> $y \in (0,x)$ <u>have property</u> $\mathcal{a}$.

(2) $[0, x]$ <u>is a nil ideal of</u> G <u>provided all</u> $y \in (0, x]$ <u>have property</u> $\mathcal{a}$ <u>and</u> $x \neq u$.

(3) <u>the radical of</u> G <u>is open</u>.

<u>Proof</u>. (1) Let $z \in [0, x)$. If $a \neq 0$, $a[0,x) \subset [0,x)$ by virtue of Theorem 3.1 and Corollary 3.2. If $a = 0$, this result is trivial. Similarly, $[0, x)a \subset [0,x)$ for all $a \in G$. Thus, $[0, x)$ is a two-sided ideal of G; and $[0, x)$ is a nil ideal by Corollary 3.2.

(2) By Lemma 3.10 $[0, x]$ is an ideal. If $x \neq u$, $[0, x]$ is a

nil ideal by Corollary 3.2.

(3) Let R be the radical of G and let N be the set of nilpotents of G. If $a \in R$, one may find a neighborhood $V(a)$ of a such that $V(a) \subset N$ and $GV(a) \subset N$ as in the proof of Theorem 4.10. Then $L^* = GV(a) \cup V(a)$ is a left nil ideal of G by virtue of Lemma 3.14. Hence, $L^* \subset R$ and $a \in V(a) \subset R$. Thus, R is open.

We note that if every element of G has property $\mathscr{a}$, G is an N-semigroup.

Definition 4.12. A topological groupoid G with a zero is said to have *arbitrarily small ideal neighborhoods of* 0 iff every neighborhood U of 0 contains an ideal which is a neighborhood of 0. This is analogous to the notion of arbitrarily small subgroups [13, Chapter 3].

Theorem 4.13. *If* G *is a compact groupoid with a zero and a unit which is irreducibly connected between* 0 *and* u *and which obeys the cancellation law, then* G *has arbitrarily small bicompact ideal neighborhoods of* 0.

Proof. The result follows immediately from Lemma 3.10.

Quasi Groups

Definition 4.13. A groupoid G is called a *quasi group* iff for any two elements a and b in G, there exists exactly one $x \in G$ such that $ax = b$ and exactly one $y \in G$ such that $ya = b$ [7, p. 986].

Definition 4.14. A groupoid G is called a *left hand quasi group* iff for any two elements a and b in G, there exists exactly one $x \in G$ such that $ax = b$ [7, p. 986].

Definition 4.15. A groupoid G is called a *right hand quasi group* iff for any two elements a and b in G, there exists exactly one

$y \in G$ such that $ya = b$ [7].

Definition 4.16. A groupoid G is said to satisfy the left cancellation law iff $ax = ay$ implies $x = y$ for all x, y, and $a \in G$; and G satisfies the right cancellation law iff $xa = ya$ implies $x = y$ for all x, y, and $a \in G$. G satisfies the cancellation law iff it satisfies the right cancellation law and the left cancellation law.

We now state a well known theorem from the theory of quasi groups.

Theorem 4.14. *A finite cancellation groupoid is a quasi group.* [7]

Under certain additional hypotheses, we will extend the theorem to bicompact groupoids.

Lemma 4.4. *Let G be a topological groupoid and B^* be a bicompact subset of G. Let $W = (\omega)$ be an index system and let $A = \{a_\omega : a_\omega \in G$ and $\omega \in W\}$ and $B = \{b_\omega : b_\omega \in G$ and $\omega \in W\}$ be subsets of G whose elements correspond to the same index system W. We suppose $B \subset B^*$ and $a \in \bar{A}$. Then, there exists $b \in \bar{B}$ such that $ab \in \bar{C}$ where* $C = \{a_\omega b_\omega : \omega \in W\}$.

Proof. Let $\wedge = \{V_t(a) : t \in T\}$ be a complete neighborhood system of a. Let $A_t = V_t(a) \cap A$. Clearly, $A_t \neq \square$. By B_t, we denote the set of elements of B whose elements have the same indices with those of elements of A_t. Let $B = \{B_t : t \in T\}$. We will show that B is a family of subsets of B^* with the finite intersection property. For let $B_{t_1}, B_{t_2}, \ldots, B_{t_n}$ be any finite number of sets in B. Then $A_{t_1}, A_{t_2}, \ldots, A_{t_n}$ are the corresponding subsets of A. Hence, there exists a neighborhood $V_{t_0}(a)$ $(\in \wedge,\ t_0 \in T)$ such that $V_{t_0}(a) \subset \bigcap_{i=1}^{n} V_{t_i}(a)$. Let $A_{t_0} = V_{t_0}(a) \cap A$ and denote by B_{t_0} the subset of B which corresponds to A_{t_0}. Then, it is clear that $\square \neq B_{t_0} \subset \bigcap_{i=1}^{n} B_{t_i}$.

Thus B has the finite intersection property, and since B^* is bicompact we have $\bigcap_{t\varepsilon T}\overline{B}_t \neq \square$. Let $b \bigcap \bigcap_{t\varepsilon T}\overline{B}_t$ and let $V(ab)$ be an arbitrary neighborhood of ab. Then there exist neighborhoods $V_k(a)$ of a and $V(b)$ of b such that $V_k(a)V(b) \subset V(ab)$ where $V_k(a)$ $(\in \wedge, k \,\varepsilon\, T)$. Since $b \,\varepsilon\, \bigcap_{t\varepsilon T}\overline{B}_t$, $V(b) \bigcap B_k \neq \square$. Now, if we let $b_{k_o} \in B_k \bigcap V(b)$ there exists an element a_{k_o} such that $a_{k_o} \in A_k = A \bigcap V_k(a)$. Hence $a_{k_o}, b_{k_o} \,\varepsilon\, V(ab) \bigcap C$ and $ab \,\varepsilon\, \overline{C}$.

Lemma 4.5. If G is a bicompact groupoid in which every element has property $\mathcal{a}$ and which satisfies the left cancellation law, then G is a left hand quasi group provided $p(\widehat{p^n}G) = (p\widehat{p^n})G = \widehat{p^n}(pG)$ for all $p \,\varepsilon\, G$ and for all $n \,\varepsilon\, I$.

Proof. If $p \in G$, it follows that $G \supset pG \supset p^2G \supset \ldots \supset p^nG \supset \ldots$. Put $P_r = \left\{p^i : i \geq r\right\}$. Then, since G is bicompact $\bigcap_{r=1}^{\infty}\overline{P_r} \neq \square$. Let $q \,\varepsilon\, \bigcap_{r=1}^{\infty}\overline{P_r}$. We intend to show that $\bigcap_{i=1}^{\infty}\overline{p^iG} = qG$.

First, we show $qG \subset \bigcap_{i=1}^{\infty}\overline{p^iG}$. Let $qx (x \in G)$ be any element of qG, and let $V(qx)$ be an arbitrary neighborhood of qx. Then one can find a neighborhood $V(q)$ of q such that $V(q)x \subset V(qx)$. But, $p^{r+i_r} \in V(q)$ for $r = 1, 2, \ldots$ where i_r is a non-negative integer. Hence, $p^{r+i_r}x \subset V(qx)$ for $r = 1, 2, \ldots$. But, $p^{r+i_r}x \in p^{r+i_r}G \supset p^rG$ for $r = 1, 2, \ldots$. Therefore, $qx \in \bigcap_{r=1}^{\infty}\overline{p^rG} = \bigcap_{r=1}^{\infty}\overline{p^rG}$ and $qG \subset \bigcap_{i=1}^{\infty}\overline{p^iG}$. Next we will show that $\bigcap_{i=1}^{\infty}\overline{p^iG} \subset qG$. Let $p' \in \bigcap_{i=1}^{\infty}\overline{p^iG}$. Then $p' = p^ig_i$, $i = 1, 2, \ldots$, where $g_i \in G$. Let $G' = \left\{g_i : i \,\varepsilon\, I\right\}$ and $P = \left\{p^i : i \,\varepsilon\, I\right\}$. By Lemma 4.4, there is a $g \in \overline{G'} \subset G$ such that $qg \in \overline{\left\{p^ig_i : i \,\varepsilon\, I\right\}} = \overline{p'} = p'$. Hence, $p' = qg \,\varepsilon\, qG$ and $\bigcap_{i=1}^{\infty}\overline{p^iG} \subset qG$. Thus, $qG = \bigcap_{i=1}^{\infty}\overline{p^iG}$. Next, we will show that $\bigcap_{i=1}^{\infty}\overline{p^i(pG)} = q(pG)$. As before, let $qx \in q(pG)$ $(x \in pG)$ and let $V(qx)$ be an arbitrary neighborhood of qx. Then, one can find a neighborhood $V(q)$ of q such that $V(q)x \subset V(qx)$. Thus, $p^{r+i_r}x \in V(qx)$ for $r = 1, 2, \ldots$. Since $p^n(pG) = p^{n+1}G$ for all positive integers n, $p^{r+i_r}x \in p^{r+i_r}(pG) = p^{r+i_r+1}G \subset p^{r+1}G = p^r(pG)$ for $r = 1, 2, \ldots$. Thus $qx \in \bigcap_{r=1}^{\infty}\overline{p^r(pG)} = \bigcap_{r=1}^{\infty}p^r(pG)$ and $q(pG) \subset \bigcap_{i=1}^{\infty}p^i(pG)$. Finally, we show that $\bigcap_{i=1}^{\infty}p^i(pG) \subset q(pG)$. Let $p' \in \bigcap_{i=1}^{\infty}p^i(pG)$. Then $p' = p^i g_i'$, $i = 1, 2, \ldots$, where $g_i' \in pG$. Let $P = \{p^i : i \in I\}$ and $G' = \{g_i' : i \in I\}$. Then there exists $g' \in \overline{G'} \subset \overline{pG} = pG$ such that $qg' \in \overline{\{p^i g_i' : i \in I\}} = \overline{p'} = p'$. Thus, $p' = qg' \in q(pG)$ and $\bigcap_{i=1}^{\infty}p^i(pG) \subset q(pG)$. Hence, $\bigcap_{i=1}^{\infty}p^i(pG) = q(pG)$. But, $\bigcap_{i=1}^{\infty}p^i(pG) = \bigcap_{i=1}^{\infty}p^{i+1}G = \bigcap_{i=1}^{\infty}p^iG$. Thus, $q(pG) = qG$ and $pG = G$ by virtue of the left cancellation law. Thus, if we are given any a and b in G, there exists $x \in G$ such that $ax = b$. By virtue of the left cancellation law, this x is unique and G is a left hand quasi-group.

Lemma 4.6. _If G is a bicompact groupoid in which every element has property $\mathcal{A}$ and which satisfies the right cancellation law, then G is a right hand quasi group provided_ $(G\hat{p}^n)p = G(\hat{p}^n p) = (Gp)\hat{p}^n$ _for all_ $p \in G$ _and for all_ $n \in I$.

Proof. The proof of this lemma is analogous to the proof of the previous lemma.

Theorem 4.15. _If G is a bicompact groupoid in which every element has property $\mathcal{A}$ and which satisfies the cancellation law, then G is a quasi group provided_ $(G\hat{p}^n)p = (Gp)\hat{p}^n = G(p\hat{p}^n)$ _and_ $p(\hat{p}^n G) = \hat{p}^n(pG) = (p\hat{p}^n)G$ _for all_ $p \in G$ and for all $n \in I$.

Proof. This result follows immediately from Lemma 4.5 and Lemma 4.6.

Example 4.6. Consider the following multiplication table

	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

Under this multiplication $G = \{1, 2, 3\}$ is a finite groupoid which satisfies the cancellation law, in which every element has property $\mathcal{a}$, and in which $(G\hat{p^n})p = (Gp)\hat{p^n} = G(p\hat{p^n})$ and $p(\hat{p^n}G) = \hat{p^n}(pG) = (p\hat{p^n})G$ for all $p \in G$ and all $n \varepsilon I$. But, G is not a semi-group.

Example 4.7. Let E^2 be the Euclidean plane with the usual topology. Define a multiplication on E^2 by defining $x \circ y$ = midpoint of the segment xy. Under this multiplication E^2 becomes a topological groupoid which is not a semi-group although it is a quasi group.

Example 4.8. Let C be the circumference of a circle in E^2 with the usual relative topology. On C define $x \circ y = z$ if z is the mirror image of y in the diameter through x. Under this multiplication C becomes a topological groupoid which is not a semi-group although it is a quasi-group.

Example 4.9. Let P be the parabola $(y = x^2)$ in E^2 with the usual relative topology. On P, define $x \circ y = z$ by the demand that $z \in P$ and that the line yz is parallel to the tangent at x (set $(xx = x)$. Under this multiplication P becomes a topological groupoid which is not a semi-group although it is a quasi group.

For the algebraic counterparts of these examples, see [22, p. 224].

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